First

M.Math.IInd year
Semestral Exam 2012
Number Theory
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Q 1. Show that if n divides $2^n - 1$, then $n = 1$.
Hint: Look at a suitable prime dividing n if $n > 1$.

Q 2. Let g be a multiplicative function and $f(n) = \sum_{d|n} g(d)$. Consider the $n \times n$ matrix A where $a_{ij} = f(GCD(i,j))$. Show that $\det A = g(1)g(2)\cdots g(n)$.
Hint: Look at the matrix B with $b_{ij} = \sqrt{g(j)}$ when $j|i$ and $b_{ij} = 0$ if not. Relate A to B.

Q 3. Let m be a product of primes of the form $4t+1$ and let n be an arbitrary integer. Prove that $y^2 = x^3 + (4n-1)^3 - 4m^2$ has no integral solutions.
Hint: Observe that any solution (x,y) satisfies $x \equiv 1 \bmod 4$; then rewrite the equality as $y^2 + 4m^2 = x^3 + (4n-1)^3$ and show that the right side must have a prime factor $\equiv 3 \bmod 4$ and derive a contradiction using quadratic reciprocity.

Q 4. Prove that the quadratic form $7x^2 + 25xy + 23y^2$ takes the same values as the quadratic form $x^2 + xy + 5y^2$ over integers.

Q 5. Find all primes p such that $x^2 \equiv 13 \bmod p$ has a solution.

Q 6. Let f be a positive-definite integral, binary quadratic form. Then, show that there are only a finite number of representations of an integer n by f.

Q 7. Prove that the sum of the primitive roots mod p is $\mu(p-1)$ mod p.

Q 8. Prove that $x^2 + y^2 + z^2 = 2xyz$ has no integer solutions $x, y, z \neq 0$.

Q 9. Prove that $\theta(x) = \psi(x) + O(\sqrt{x})$ where $\theta(x) := \sum_{p \leq x} \log(p)$ and $\psi(x) := \sum_{n \leq x} \Lambda(n)$ with $\Lambda(n)$ the Mangoldt function.